Graphical House Allocation

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Graphical House Allocations

Given n *agents* N, n *houses* H, *identical valuations* $v : H \to \mathbb{R}_{\geq 0}$, and an *undirected graph* $G = (N, E)$, output a complete allocation minimizing the total envy over edges of G. In other words, find an allocation π^* that minimizes $\sum_{(i,j)\in E} |v(h_i) - v(h_j)|$.

We assume that $H = \{h_1, \ldots, h_n\}$, with $v(h_1) \leq \ldots \leq v(h_n)$.

The values of H can be represented as marks on an interval of the real numbers. Any allocation π can be represented by means of line segments under the corresponding subintervals, one for each edge of G. We wish to find the allocation that minimizes the sum of the lengths of these line segments.

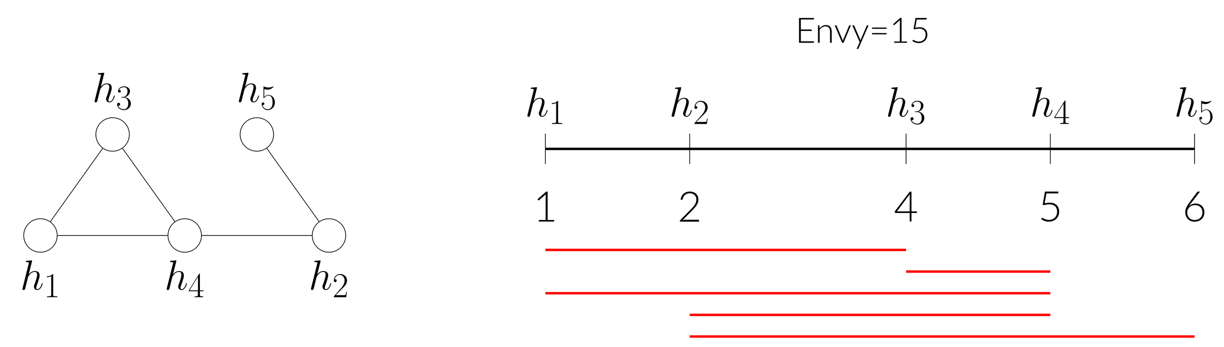


Figure 1. (Left) Graph on five agents with an allocation π. (Right) Envy along the edges of G displayed in red as separate line segments. The total envy is 15.

Motivation

- Graphical House Allocation is a **natural** constraint, as stakeholders only care about the ones they interact with.
- We want to get **structural insights** into graphs to understand the algorithmic problem at its fundamentals.

Note: The problem of minimum-envy house allocation on a graph is similar to the work of Beynier et al. [2019], though we restrict to identical valuations and investigate more classes of graphs. Our problem is the classical problem when G is the complete graph K_n.

Linear Arrangements

The classical **minimum linear arrangement problem** (MLA) asks the same question, when the values (the range of the function v) are equally spaced on the interval, i.e. WLOG they form the set $\{1, \ldots, n\}$.

It is known that MLA is NP-hard on general graphs and even bipartite graphs [Garey et al., 1976]. However, MLA on trees, forests, and many other simple classes of graphs is polynomial time [Chung, 1984].

Hardness

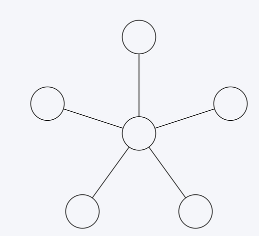
Theorem. The graphical minimum-envy house allocation problem is NP-complete, even when the values are in $\{0, 1\}$.

Pf. There is a straightforward reduction from the Minimum Bisection problem on a graph G to the minimum-envy house allocation problem on G with the values in $\{0, 1\}$.

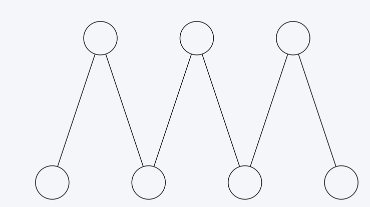
Corollary. Even when valuations are in the interval $[0, 1]$, the graphical minimum-envy house allocation problem is hard to approximate within an additive factor of $n^{2-\epsilon}$ for any $\epsilon > 0$.

Connected Graphs

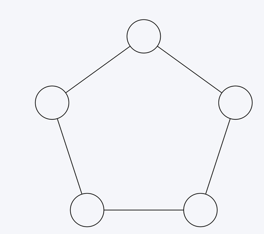
Theorem. If G is the **star** $K_{1,n}$, then the minimum-envy allocation π^* under identical valuations assigns a median of the values in the center of the star, and the rest on the spokes in any order.



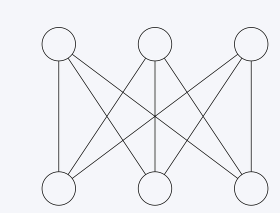
Theorem. If G is the **path** P_n, then the minimum-envy allocation π^* under identical valuations attains a total envy of $v(h_n) - v(h_1)$, and places the houses in sorted order along P_n.



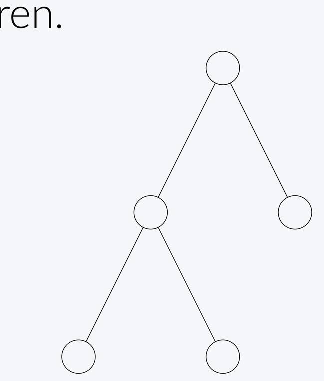
Theorem. If G is the **cycle** C_n, then any minimum-envy allocation π^* under identical valuations attains a total envy of $2(v(h_n) - v(h_1))$, and corresponds to placing h_1 and h_n arbitrarily on any two vertices of the cycle, and the remaining houses so that each of the two paths from h_1 to h_n along the cycle consists of houses in sorted order.



Theorem. If G is the **complete bipartite graph** $K_{r,s}$ ($r \geq s$), the minimum-envy allocation π^* allocates the least valued and highest valued $(r - s)/2$ houses to the larger side, and the remaining values (roughly) alternately to the two sides.



Theorem. If G is a **rooted binary tree** T, at least one minimum-envy allocation satisfies the *local median property*: every internal node of the tree receives a value that is the median of the values on the node and its two children.



Conjecture. If G is a rooted binary tree T, at least one minimum-envy allocation satisfies the *global median property*: every internal node of the tree receives a value that is the median of the values on the node and its two *subtrees*. This would imply a $O(2^d)$ algorithm for rooted binary trees, where d is the maximum depth.

Acknowledgments

We would like to thank Andrew McGregor, Cameron Musco, and Yair Zick for immensely helpful discussions and feedback.

Disconnected Graphs

MLA and minimum-envy house allocation are **completely different** on disconnected graphs!

Theorem (Seidvasser [1970]). In any graph G, MLA assigns contiguous intervals of values to each connected component of G.

This means that MLA is solvable in poly-time when each component is solvable in poly-time.

Theorem (NP-Hardness). The minimum-envy house allocation is NP-hard even on disjoint unions of paths (or cycles, or stars).

Proof. Reduction from Unary Bin-Packing.

Note that in MLA, forests have $O(n^{\log_2 3})$ time algorithms (Chung [1984])!

Separability

We can use the idea of *separability* to construct FPT algorithms for some disconnected graphs.

Definition (Separability). Let G be disconnected with connected components $C_1, \ldots, C_k$.

1. G is **separable** if for any instance, there is an optimal allocation where some C_i receives a contiguous set of values, subject to which C_j receives a contiguous set of values, and so on.
2. G is **strongly separable** if for any instance, some optimal allocation assigns contiguous subsets of values to each C_i.
3. G is **inseparable** if there is an instance where every optimal allocation assigns non-contiguous values to two (or more) components.

Important Observation: Strongly separable graphs have FPT algorithms and separable graphs have XP algorithms, in the number of connected components (if each component is solvable in poly-time).

By Seidvasser [1970], in MLA, every graph is strongly separable. In the minimum-envy house allocation problem, however, there are inseparable graphs (including inseparable forests), and also separable graphs that are not strongly separable.

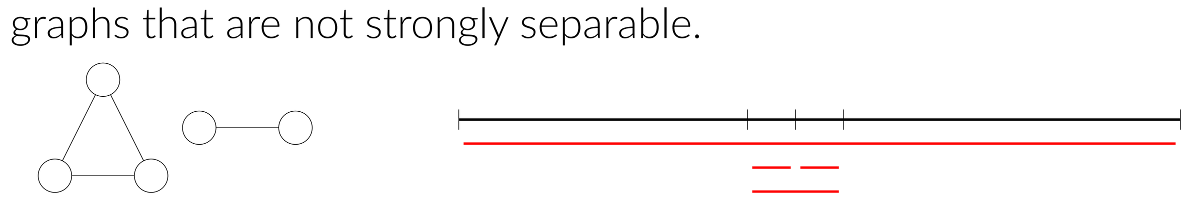


Figure 2. For the instance above, the optimal allocation to $P_2 + C_3$ is to give the two extreme-valued houses to the edge, and the cluster in the middle to the triangle. This graph is separable but not strongly separable.

Classification of Disconnected Graphs

Theorem. Arbitrary disjoint unions of paths (or cycles, or stars) are **strongly separable**.

Theorem. An arbitrary disjoint union of cliques $K_{n_1}, K_{n_2}, \ldots K_{n_r}$ is **separable**. Furthermore, it is **strongly separable** if and only if all the cliques are equisized.

Theorem. There is an **inseparable** forest.

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